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Approximation on Compact Riemannian Globally Symmetric Manifolds of Rank One

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1. INTRODUCTION

Throughout this paper, M will denote a compact Riemannian globally symmetric manifold of rank one [14]. Let the group $\mathrm{Iso}(M)$ of all isometries of M be equipped with its natural Lie group structure [17]. If G denotes the neutral connected component of $\mathrm{Iso}(M)$ and K denotes the stabilizer of some base point $\mathbb{1} \in M$, then K is a closed subgroup of the compact connected Lie group G and M is diffeomorphic to the left coset space G/K. The Riemannian symmetric pair (G, K) is either of the Euclidean type or of the compact type. In the first case, M is diffeomorphic to the one-dimensional torus group $\mathbb{T}$. Since we are interested only in the case $\dim_{\mathbb{R}} M > 1$, we shall suppose that G is a semisimple compact connected Lie group.

Let μ be the normalized Haar measure of G and $\kappa : G \to M$ the canonical surjection. Then $\nu = \kappa(\mu)$ is the normalized G-invariant Radon measure on M. It is well known that (G, K) forms a Gelfand pair [10, 18]. Thus K is a massive subgroup of G. Let $\hat{M}$ denote the unitary dual of M, i.e., the set of all equivalence classes of continuous irreducible unitary representations of G that are of class 1 with respect to K [7, 8, 24]. For each $\lambda \in \hat{M}$ there exists a complex Hilbert space $\mathscr{H}_\lambda(M)$ of finite dimension d_λ that is invariant with respect to the action of G on M such that $L_{\mathbb{C}}^2(M; \nu)$ admits the Hilbert sum decomposition

$$L_{\mathbb{C}}^2(M; \nu) = \bigoplus_{\lambda \in \hat{M}} \mathscr{H}_\lambda(M)$$

corresponding to the Peter–Weyl decomposition of the complex Hilbert space $L_{\mathbb{C}}^2(G; \mu)$. The spaces $(\mathscr{H}_\lambda(M))_{\lambda \in \hat{M}}$ consist of continuous complex-valued functions on M and are minimal G-invariant vector subspaces of $L_{\mathbb{C}}^2(M; \nu)$.

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Let $\mathscr{J}$ denote a grouping of $\hat{M}$, i.e., a family of finite subsets of $\hat{M}$ that are strictly increasing with respect to inclusion and cover the whole of $\hat{M}$. For any $J \in \mathscr{J}$ let $\mathscr{H}_J(M) = \bigoplus_{\lambda \in J} \mathscr{H}_\lambda(M)$. Moreover, let

$$Q_J : \mathscr{C}_{\mathbb{C}}(M) \to \mathscr{H}_J(M) \tag{1}$$

be a continuous linear projector, i.e., a continuous idempotent linear mapping of the complex Banach space $\mathscr{C}_{\mathbb{C}}(M)$ onto its vector subspace $\mathscr{H}_J(M)$.

In the present paper we ask the following question. Given $\mathscr{J}$, does the family $(Q_J)_{J \in \mathscr{J}}$ define a convergent approximation process in $\mathscr{C}_{\mathbb{C}}(M)$? In other words: Does the (uniform) convergence

$$\lim_{J \in \mathscr{J}} \| Q_J f - f \|_\infty = 0$$

hold for each function $f \in \mathscr{C}_{\mathbb{C}}(M)$? A negative answer will be given in Theorem 4 of Section 4. The proof is based on the classification of compact Riemannian globally symmetric manifolds of rank one that is outlined for the reader's convenience in Section 2. It proceeds in several stages that are of some interest for their own sake (Sections 3 and 4). Section 5 deals with the case $M = \mathbb{S}_3$ in view of the fact that $\mathbb{S}_3$ assumes a distinguished rôle among the compact spheres $(\mathbb{S}_n)_{n \geqslant 2}$. Finally, Section 6 is concerned with some additional comments.

2. Compact Riemannian Globally Symmetric Manifolds of Rank One

Keeping to the notations of the preceding section, let $\mathscr{M}_{\mathbb{C}}(M) = \mathscr{C}_{\mathbb{C}}'(M)$ be the vector space of all complex Radon measures on M and $\langle \cdot, \cdot \rangle$ the canonical bilinear form associated with the duality $(\mathscr{C}_{\mathbb{C}}(M), \mathscr{M}_{\mathbb{C}}(M))$. For each class $\lambda \in \hat{M}$ let $H_\lambda \geqslant 0$ be the Schwartz kernel of $\mathscr{H}_\lambda(M) \in \operatorname{Hilb}(\mathscr{C}_{\mathbb{C}}(M))$ with respect to the Banach space $\mathscr{C}_{\mathbb{C}}(M)$ [22]. If $\epsilon_x \in \mathscr{M}_{\mathbb{C}}(M)$ denotes the Dirac measure located at the point $x \in M$ then the zonal K-spherical function ${}_y\omega_\lambda \in \mathscr{H}_\lambda(M)$ of positive type with pole $y \in M$ is given by

$${}_y\omega_\lambda : M \ni x \rightsquigarrow \langle H_\lambda \epsilon_y , \epsilon_x \rangle \in \mathbb{C}.$$

For each $J \in \mathscr{J}$ the Fourier projector $P_J : \mathscr{C}_{\mathbb{C}}(M) \to \mathscr{H}_J(M)$ takes the form

$$P_J : f \rightsquigarrow \left(M \ni x \rightsquigarrow \sum_{\lambda \in J} \int_M {}_y\omega_\lambda(x) f(y)\, d\nu(y)\right). \tag{2}$$

To determine explicitly the functions $({}_1\omega_\lambda)_{\lambda \in \hat{M}}$ it should be observed that M is a compact two-point homogeneous space. According to the classification of these spaces [25] the following list of compact Riemannian globally symmetric manifolds of rank one and real dimension $n > 1$ is complete [15]:

(i) $M = \mathbb{S}_n$, the spheres of dimension $n \geqslant 2$;

(ii) $M = \mathbb{P}_n(\mathbb{R})$, the real projective spaces of dimension $n \geqslant 2$;

(iii) $M = \mathbb{P}_n(\mathbb{C})$, the complex projective spaces of real dimension $n \geqslant 4$, $n \in 2\mathbb{N}$;

(iv) $M = \mathbb{P}_n(\mathbb{H})$, the quaternionic projective spaces of real dimension $n \geqslant 8$, $n \in 4\mathbb{N}$;

(v) $M = \mathbb{P}$ (Cay), the Cayley elliptic plane of real dimension 16.

The Riemannian symmetric pairs (G, K) such that M is diffeomorphic to G/K are as follows:

(i) $G = SO(n+1)$ $\quad K = SO(n), n \geqslant 2$;

(ii) $G = SO(n+1)$ $\quad K = O(n), n \geqslant 2$;

(iii) $G = SU(\frac{1}{2}n+1)$ $\quad K = S(U(\frac{1}{2}n) \times U(1)), n \geqslant 4, n \in 2\mathbb{N}$;

(iv) $G = Sp(\frac{1}{4}n+1)$ $\quad K = Sp(\frac{1}{4}n) \times Sp(1), n \geqslant 8, n \in 4\mathbb{N}$;

(v) $G = F_{4(-52)}$ $\quad K = SO(9)$.

All the geodesics in M are simply closed and have the same length 2ℓ where $\ell > 0$ denotes the diameter of M. Let $\mathscr{ZC}\mathscr{C}_{\mathbb{C}}(M)$ be the closed vector subspace of $\mathscr{C}_{\mathbb{C}}(M)$ formed by the zonal functions $f \in \mathscr{C}_{\mathbb{C}}(M)$. Clearly, $\mathscr{ZC}\mathscr{C}_{\mathbb{C}}(M)$ may be identified with the complex Banach space $\mathscr{C}_{\mathbb{C}}(K\backslash G/K)$ and $f \in \mathscr{C}_{\mathbb{C}}(M)$ belongs to $\mathscr{ZC}\mathscr{C}_{\mathbb{C}}(M)$ if and only if f depends only on the distance $\theta_x \in [0, \ell]$ of its argument $x \in M$ from the base point $\mathbb{1}$. More precisely, we have $f \in \mathscr{ZC}\mathscr{C}_{\mathbb{C}}(M)$ if and only if there exists a function $f^\natural \in \mathscr{C}_{\mathbb{C}}([-1, 1])$ such that the identity

$$f(x) = f^\natural(\cos 2\ell_0\theta_x)$$

holds for all points $x \in M$ with a suitable number $\ell_0 > 0$ depending upon the metric of M. The mapping $f \rightsquigarrow f^\natural$ defines an isomorphism of $\mathscr{ZC}\mathscr{C}_{\mathbb{C}}(M)$ onto $\mathscr{C}_{\mathbb{C}}([-1, +1])$.

The zonal K-spherical functions $({}_{\mathbb{1}}\omega_\lambda)_{\lambda \in \hat{M}}$ of positive type on M with pole $\mathbb{1}$ are eigenfunctions of the Laplace–Beltrami operator Δ of M. If we choose a geodesic polar coordinate system with pole at the base point $\mathbb{1}$ and colatitude θ, the radial part of Δ takes the Jacobi operator form

$$\Delta_\theta = \frac{1}{\sin^p \ell_0\theta \cdot \sin^q 2\ell_0\theta} \cdot \frac{d}{d\theta}\left(\sin^p \ell_0\theta \cdot \sin^q 2\ell_0\theta \cdot \frac{d}{d\theta}\right).$$

The multiplicities $p \geqslant 0$ and $q \geqslant 0$ are determined by the structure of the Lie algebras of G and K [4]. We quote the following list from Helgason [15]:

(i)	$M = \mathbb{S}_n$	$p = 0$	$q = n - 1$	$\ell_0 = \pi/2\ell, n \geqslant 2;$
(ii)	$M = \mathbb{P}_n(\mathbb{R})$	$p = 0$	$q = n - 1$	$\ell_0 = \pi/4\ell, n \geqslant 2;$
(iii)	$M = \mathbb{P}_n(\mathbb{C})$	$p = n - 2$	$q = 1$	$\ell_0 = \pi/2\ell, n \geqslant 4, n \in 2\mathbb{N};$
(iv)	$M = \mathbb{P}_n(\mathbb{H})$	$p = n - 4$	$q = 3$	$\ell_0 = \pi/2\ell, n \geqslant 8, n \in 4\mathbb{N};$
(v)	$M = \mathbb{P}(\mathrm{Cay})$	$p = 8$	$q = 7$	$\ell_0 = \pi/2\ell.$

If $\hat{M}$ is identified with a subset of $\mathbb{N}$ by means of the natural bijection and $P_m^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree m with indices

$$\alpha = \tfrac{1}{2}(p + q - 1), \qquad \beta = \tfrac{1}{2}(q - 1)$$

and standardization $P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}$ then we have the list:

(i)	$M = \mathbb{S}_n$	${}_{\natural}\omega_m{}^{\natural} : \theta \rightsquigarrow a_m P_m^{(\alpha,\beta)}(\cos 2\ell_0\theta), \quad m \in \mathbb{N};$
(ii)	$M = \mathbb{P}_n(\mathbb{R})$	${}_{\natural}\omega_m{}^{\natural} : \theta \rightsquigarrow a_m P_m^{(\alpha,\beta)}(\cos 2\ell_0\theta), \quad m \in 2\mathbb{N};$
(iii)	$M = \mathbb{P}_n(\mathbb{C})$	${}_{\natural}\omega_m{}^{\natural} : \theta \rightsquigarrow a_m P_m^{(\alpha,\beta)}(\cos 2\ell_0\theta), \quad m \in \mathbb{N};$
(iv)	$M = \mathbb{P}_n(\mathbb{H})$	${}_{\natural}\omega_m{}^{\natural} : \theta \rightsquigarrow a_m P_m^{(\alpha,\beta)}(\cos 2\ell_0\theta), \quad m \in \mathbb{N};$
(v)	$M = \mathbb{P}(\mathrm{Cay})$	${}_{\natural}\omega_m{}^{\natural} : \theta \rightsquigarrow a_m P_m^{(\alpha,\beta)}(\cos 2\ell_0\theta), \quad m \in \mathbb{N}.$

Here $(a_m)_{m\geqslant 0}$ denotes a sequence of standardization constants.

3. Preliminary Results

Let $\mathscr{C}_{\mathbb{C}}{}^b(\mathbb{R})$ be the vector space of all bounded continuous complex-valued functions on the real line $\mathbb{R}$ equipped with the Čebyšev norm $\|\cdot\|_\infty$. Denote by E the closed vector subspace of the complex Banach space $\mathscr{C}_{\mathbb{C}}{}^b(\mathbb{R})$ formed by all functions $f \in \mathscr{C}_{\mathbb{C}}{}^b(\mathbb{R})$ that are even and have the period 2π. Moreover, for an arbitrary finite subset J_0 of $\mathbb{N}$ such that $0 \in J_0$ let E_{J_0} be the vector subspace of E spanned by the even trigonometric monomials

$$\{\theta \rightsquigarrow \cos n\theta \mid n \in J_0\}$$

and

$$D_{J_0} : \mathbb{T} \ni e^{it} \rightsquigarrow \sum_{\substack{n\in\mathbb{Z}\\ |n|\in J_0}} e^{int}$$

the Dirichlet kernel associated with J_0.

THEOREM 1. *Let $L_{J_0} : E \to E$ be a continuous endomorphism of the complex Banach space E that is idempotent and satisfies $L_{J_0}(E) = E_{J_0}$. Then the estimate*

$$\| id_E - L_{J_0} \| \geqslant \tfrac{1}{2}(1 + \| D_{J_0'} \|_1)$$

holds.

Proof. Identify E in the natural way with a closed vector subspace of the complex Banach space $\mathscr{C}_{\mathbb{C}}(T)$ and let $F_{J_0} : E \ni f \rightsquigarrow f * D_{J_0} \in E$ be the Fourier projector associated with J_0. By an adaptation of the reasoning used in Cheney [5, Chap. 6], in the special case $J_0 = \{1, 2, ..., N\}$ we obtain via a symmetrization formula the estimate

$$2 \| id_E - L_{J_0} \| \geq \| id_E - F_{J_0} \| = 1 + \| D_{J_0} \|_1 ,$$

as contended.

THEOREM 2. *Suppose that* $0 < k < 6^{-1/2}(1 - e^{-2})$ *and let* $N = \mathrm{Card}(J_0)$ *be sufficiently large depending upon* k. *If the mapping* $L_{J_0} : E \to E$ *satisfies the hypotheses of Theorem 1 then the inequality*

$$\| id_E - L_{J_0} \| \geq \tfrac{1}{2} \left(1 + k \left(\frac{\log N}{\log \log N}\right)^{1/2}\right)$$

obtains.

Proof. Combine Theorem 1 with the Cohen–Davenport–Pichorides theorem ([6, 9]; also see [1, 16]).

4. DIVERGENCE THEOREMS

By virtue of the results obtained in the preceding section we are now in a position to prove the following divergence result.

THEOREM 3. *Let* M *denote a compact Riemannian globally symmetric manifold of rank one. Suppose that* $\mathscr{J}$ *is a grouping of its unitary dual* $\hat{M} \subseteq \mathbb{N}$ *such that* $0 \in J$ *for all sets* $J \in \mathscr{J}$ *and* $S = \{x_j \mid j \in \mathbb{N}\}$ *is a preassigned sequence of distinct points belonging to* M. *There exists a fat subset* F *of the complex Banach space* $\mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(M)$ *such that*

$$\lim_{J \in \mathscr{J}} P_J f(x) = +\infty$$

for all functions $f \in F$ *and all points* $x \in S$.

Proof. We shall consider the case (i) in the classification of the compact Riemannian globally symmetric manifolds of rank one. Thus $M = \mathbb{S}_n$, $n \geq 2$. We have $\hat{M} = \mathbb{N}$. Fix the north pole $\mathbb{1} = (0, ..., 0, 1)$ of $\mathbb{S}_n$ as the base point. For each $m \in \mathbb{N}$, $\mathscr{H}_m(\mathbb{S}_n)$ is the complex vector space of all surface spherical harmonics of degree m on $\mathbb{S}_n$, i.e., the vector space of the restrictions to $\mathbb{S}_n$ of all harmonic homogeneous polynomials of degree m in $n + 1$

real variables and complex coefficients. The complex dimension of $\mathscr{H}_m(\mathbb{S}_n)$ is given by

$$d_m = \frac{(n+m-2)!\,(n+2m-1)}{(n-1)!\,m!} \qquad (m \in \mathbb{N}).$$

Furthermore, ω_m is the zonal spherical harmonic of degree m with pole $\mathbb{1}$ ([3] or [19, Part II, Chap. III]). Since $\ell = \pi$, $2\ell_0 = 1$ we have in the present case

$$_{\mathbb{1}}\omega_m : \theta \rightsquigarrow a_m P_m^{\left(\frac{n-2}{2}, \frac{n-2}{2}\right)}(\cos\theta) \qquad (m \in \mathbb{N}).$$

To adopt the customary notation and normalization, let $P_m^{(\lambda)}$ denote the ultraspherical (or Gegenbauer) polynomial of degree m and index $\lambda > -\frac{1}{2}$ with standardization $P_m^{(\lambda)}(1) = \binom{m+2\lambda-1}{m}$.

Put

$$b_m = \frac{n+2m-1}{n-1}.$$

Then the zonal spherical harmonics $({}_{\mathbb{1}}\omega_m)_{m\geqslant 0}$ give rise to the functions

$$_{\mathbb{1}}\omega_m{}^{\natural} : \theta \rightsquigarrow b_m P_m^{\left(\frac{n-1}{2}\right)}(\cos\theta) \qquad (m \in \mathbb{N}). \tag{3}$$

Let the sequence $(c_m)_{m\geqslant 0}$ form the spectrum of the complex commutative Banach algebra $\mathscr{Z}\mathscr{O}L_{\mathbb{C}}^1(M; \nu) = L_{\mathbb{C}}^1(SO(n)\backslash SO(n+1)/SO(n))$. The characters c_m take the form

$$c_m : f \rightsquigarrow (1/d_m) \int_{SO(n+1)} f(u)\, {}_{\mathbb{1}}\omega_m(u^{-1}\mathbb{1})\, d\mu(u)$$

and for each $J \in \mathscr{J}$ the restriction $L_J = P_J \mid \mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n)$ of the Fourier projector (2) admits the representation

$$L_J : \mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n) \ni f \rightsquigarrow \sum_{m\in J} c_m(f) \cdot {}_{\mathbb{1}}\omega_m \in \mathscr{Z}\mathscr{O}\mathscr{H}_J(\mathbb{S}_n).$$

From an expansion of the generating function we obtain the trigonometric representation

$$P_m^{\left(\frac{n-1}{1}\right)}(\cos\theta) = 2 \sum_{0\leqslant j\leqslant \frac{1}{2}[m]} \alpha_j \alpha_{m-j} \cos(m-2j)\,\theta$$

of the ultraspherical polynomials occurring in (3). The coefficients are

$$\alpha_j = \binom{j + \frac{n-1}{2} - 1}{j} \qquad (j \in \mathbb{N})$$

(see [23, Chap. IV]). Let $J_0 = \bigcup_{m \in J} \{m - 2j \mid 0 \leqslant j \leqslant \frac{1}{2}[m]\}$. The mapping

$$(\mathbb{R} \ni \theta \rightsquigarrow f^{\natural}(\cos\theta)) \rightsquigarrow (\mathbb{R} \ni \theta \rightsquigarrow (L_J f)^{\natural}(\cos\theta))$$

satisfies the hypotheses of Theorem 1. Thus, by Theorem 2, we obtain

$$\sup_{J \in \mathscr{J}} \| L_J \| = +\infty. \tag{4}$$

For each $J \in \mathbb{N}$ define the lower semicontinuous seminorm $p_j : \mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n) \to \mathbb{R}_+ \cup \{+\infty\}$ according to

$$p_j : f \rightsquigarrow \sup_{J \in \mathscr{J}} | L_J f(x_j) | .$$

Moreover, let

$$Z = \{ f \in \mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n) \mid \inf_{j \in \mathbb{N}} p_j(f) < +\infty \}$$

and suppose that Z is a nonmeager subset of the complex Banach space $\mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n)$. An application of the uniform boundedness principle entails the existence of a number $j_0 \in \mathbb{N}$ such that p_{j_0} is finite valued and continuous on $\mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n)$. Thus we have

$$\Big\| \sum_{m \in J} {}_{\natural}\omega_m \Big\|_1 < +\infty .$$

But this is contradicted by (4) showing that Z is a meagre set and the complement F of Z with respect to $\mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_n)$ is fat. The theorem is thereby established for the case (i). The proof for the cases (ii)–(v) proceeds similarly.

THEOREM 4. *Let the manifold M and the grouping $\mathscr{J}$ be as in Theorem 3. If $(Q_J)_{J \in \mathscr{J}}$ denotes a family of continuous linear projectors as in* (1) *then there exists a function $f \in \mathscr{C}_{\mathbb{C}}(M)$ such that the condition*

$$\sup_{J \in \mathscr{J}} \| Q_J f \|_\infty = +\infty$$

holds. Thus the approximation process $(Q_J)_{J \in \mathscr{J}}$ is divergent in $\mathscr{C}_{\mathbb{C}}(M)$.

Proof. By the preceding theorem we have

$$\sup_{J \in \mathscr{J}} \| P_J \| = +\infty.$$

The Charshiladze–Lozinski theorem for G-homogeneous Banach spaces [11, 12] implies via the Marcinkiewicz–Berman symmetrization formula that the inequality

$$\| P_J \| \leqslant \| Q_J \| \qquad (J \in \mathscr{J})$$

holds.

Thus an application of the Banach–Steinhaus theorem yields the result.

Remark 1. Theorem 4 remains valid when $(Q_J)_{J\in\mathscr{J}}$ denotes a family of continuous linear projectors of the Lebesgue space $L_{\mathbb{C}}^1(M; \nu)$ into the vector subspaces $\mathscr{H}_J(M)$, $J \in \mathscr{J}$.

Remark 2. It should be emphasized that Theorems 3 and 4 are valid for arbitrary groupings $\mathscr{J}$ of $\hat{M}$. For the case of special groupings see [11, 12]. In this connection also see [13] and the paper [21] for a survey.

5. The Case $M = \mathbb{S}_3$

Let H be a compact Lie group of positive dimension. If H acts freely on a compact sphere $\mathbb{S}_n$, $n \geqslant 1$, then H is isomorphic to a subgroup of the special unitary group $SU(2)$ (cf. [2, Chap. III]). Specifically, there are precisely three possibilities: $H = \mathbb{T}$, $H = N(\mathbb{T})$ the normalizer of $\mathbb{T}$, and $H = SU(2)$. If $SU(2)$ is considered as the compact group

$$\left\{ q = q_0 1 + \sum_{1 \leqslant k \leqslant 3} q_k j_k \in \mathbb{H} \right\} \mid q \mid = 1 \right\}$$

of unit quaternions, then $N(\mathbb{T})$ is generated by $\mathbb{T}$ and j_2 and has two connected components $\mathbb{T}$ and $j_2\mathbb{T}$. Thus the sphere $\mathbb{S}_n$ ($n > 1$) carries a Lie group structure which is compatible with its topology if and only if $n = 3$. In the case $M = \mathbb{S}_3$ the zonal spherical harmonics, the zonal $SO(3)$-spherical functions of positive type on $\mathbb{S}_3$, and the characters of $SU(2)$ coincide up to some standardization constants with the Čebyšev polynomials of the second kind. Since $\mathscr{Z}\mathscr{O}\mathscr{C}_{\mathbb{C}}(\mathbb{S}_3)$ may be identified with the center $\mathscr{Z}\mathscr{C}_{\mathbb{C}}(SU(2))$ of the complex convolution algebra $\mathscr{C}_{\mathbb{C}}(SU(2))$, Theorem 3 implies the following special result.

THEOREM 5. *Let $\mathscr{J}$ be a grouping of $\mathbb{N}$ such that $0 \in J$ for all sets $J \in \mathscr{J}$ and S a countable set of points in $SU(2)$. There exists a fat subset F of the Banach space $\mathscr{Z}\mathscr{C}_{\mathbb{C}}(SU(2))$ of all central continuous complex-valued functions on $SU(2)$ such that*

$$\lim_{J\in\mathscr{J}} P_J f(x) = +\infty$$

for all functions $f \in F$ and all points $x \in S$.

Recently Price [20] has proved this result by an extension of the Cohen–Davenport theorem to the compact unitary groups $U(2)$ and $SU(2)$.

6. Concluding Remark

The proof of the preceding results are based on the symmetrization technique (Theorems 1 and 4) and the Cohen–Davenport–Pichorides theorem.

In a forthcoming paper some additional applications of this method will be given. In particular, the problem of noncomplemented vector subspaces of Banach spaces will be investigated.

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